JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 86 (1997) 297–310

Matrices of Sinc methods[1]

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Received 15 January 1997; received in revised form 16 July 1997

Dedicated to W.B. Gragg on his 60th Birthday

Abstract

This paper gives a brief review of Sinc methods, with emphasis on the matrices of Sinc methods. A novel procedure is presented, based on Sinc convolution, for solving a Poisson problem over a rectangular region. Although some of the work of Gragg may already be applied to the solution of Sinc–matrix problems, this paper also points to new directions of matrix research.

Keywords: Sinc methods; Poisson problem; Matrix

1. Introduction and summary

Much of W.B. Gragg's mathematically beautiful work investigates the connection of Toeplitz matrices and rational approximation [3]. My area of research – Sinc methods – is a family of approximation formulas that require matrix methods for their practical implementation [14, 15, 18, 22]. Sinc methods are close to optimal approximation methods in *Sinc spaces* of functions [7, 14, 21], and in this setting, there is some intersection with Gragg's work, in as much as it has been shown [7] that Sinc and rational approximation converge at the same rate in these Sinc spaces [14, 21], and in as much as Sinc methods, which are intimately related to FFT yield systems of equations which are frequently conveniently converted to circulant matrix problems, and the work of Ammar and Gader [1] has applications to such problems. The Fourier transforms of Sinc series yield DFT formulas, and the approximation of Hilbert transforms via Sinc methods [18, Sections 5.2, 3.4 and 6.7] can involve Vandermonde matrices and Cauchy systems. For example, the leading Sinc

[1] Supported by NSF grant No. CCR-9307602.

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PII S0377-0427(97)00163-5

matrices that approximate derivatives are always Toeplitz matrices, and the even order ones are, in fact, negative definite. The work of Ammar and Gragg [4], Gragg [12], and Heinig and Rost [13] applies to these. Indeed, the matrix stemming from the Sinc approximation of the first derivative, and the Sinc matrices stemming from Hilbert transforms are, in fact, Cauchy matrices, and the methods of Boros et al. [3], and Calvetti and Reichel [8, 9] may be applied to the solution of such matrix problems. We also mention here that some of the work of Pereyra (see, e.g., [11]) has connections with Stenger's work [17]. The solution of some elliptic PDE problems via Sinc methods (see [15, 18, Section 7.4]) can also be carried out via diagonalization of matrices. The contributions of Ammar and Gragg [2, 3] can most definitely be used to speed up the algorithms for solving such problems.

We also mention that very little has been written so far about the matrices arising in Sinc methods. The little that has been done to date on this subject suggests that these matrices have properties similar to the matrices of finite difference methods. A thorough study of these matrices, such as that of the beautiful work of Varga [24] would greatly enhance the use of Sinc methods of solving partial differential and integral equations.

In the present paper we present the simple equations of Sinc approximations in both symbolic and linear algebraic forms, and we also state some unsolved matrix problems for these approximations. At the outset, we describe Sinc approximation in the setting of Sinc spaces. Then we proceed to the description of Sinc approximation.

We shall define *Sinc spaces* of approximation in Section 2.5 below. The fact that nearly all solutions of PDE and IE (partial differential and integral equation(s)) problems stemming from applications belong to Sinc spaces of approximation is not well known to solvers of such problems, most of whom, even the more mathematically adept, are accustomed to placing these solutions in Sobolev spaces, in which splines and finite elements (that go hand in hand with finite difference methods) are optimal bases for approximation, in the sense of minimizing the error in spaces of measurable functions. In addition, since the onslaught of computers, solvers of such problems have in the US been privileged with computers that are much faster than those of the rest of the world, and thus they have had no need to learn faster methods of computation. The case of a PDE or IE with the property that some higher derivative of its solution is merely measurable and not analytic seldom arises from problems stemming in applications. Scientists who derive mathematical models of physical phenomena are not accustomed to use as models functions such as Weierstrass', that are continuous but nowhere differentiable. Sinc bases form close to optimal bases in Sinc spaces of analytic functions, and as a result, Sinc methods nearly always converge more rapidly and have lower complexity than finite difference or finite element methods. Sinc methods converge at an exponential rate (i.e., at the rate $\mathcal{O}(\exp\{-n^{1/(2p)}\})$ in p dimensions) even though the solutions of the PDE or IE may have singularities on boundaries of their domain, or even if the regions on which solutions exist may be semi-infinite or infinite. Finally, we mention that Sinc programs are relatively short, and the same program based on a single integer parameter N can be used to yield arbitrary accuracy, simply by selecting N sufficiently large.

2. Sinc methods

In this section we give a complete, although brief summary of Sinc methods.

2.1. The Wiener space of approximation

Let h be a positive constant, and let f be an entire function, such that for all z in the complex plane $\mathbb{C}$, we have

$$|f(z)| \leqslant C\,e^{\pi|z|/h}, \tag{2.1}$$

with C some positive constant, and such that $f \in L^2(\mathbb{R})$. We set

$$\begin{aligned} \operatorname{sinc}(z) &= \frac{\sin(\pi z)}{\pi z}, \\ S(k,h)\circ(z) &= \operatorname{sinc}\left(\frac{z}{h} - k\right) \end{aligned} \tag{2.2}$$

with $k \in \mathbb{Z}$, the set of integers.

Theorem 2.1 (Stenger [18, Section 1.11]). *Under the above conditions, the identity*

$$f(z) = \sum_{k=-\infty}^{\infty} f(kh) S(k,h)\circ(z) \tag{2.3}$$

is valid for all $z \in \mathbb{C}$.

The space $W(\pi/h)$ of all such functions is called the *Wiener space*.

2.2. Approximation on $\mathbb{R}$

Approximation (2.3) is a convenient space of approximation on $\mathbb{R}$ for the following reasons:

1. The expansion (2.3) is replete with beautiful identities in the space $W(\pi/h)$.
2. For every $d > 0$, the expansion (2.3) and other identities obtainable from (2.3) are incredibly accurate approximations in the space $H^1(\mathscr{D}_d)$ of all functions that are analytic in the region

$$\mathscr{D}_d = \{z \in \mathbb{C}:\ |\Im z| < d\}, \tag{2.4}$$

and absolutely integrable around the boundary of $\mathscr{D}_d$, the error of approximation in each case being of the order of $\exp(-\pi d/h)$.

3. The basis $\{S(k,h)\}_{k=-\infty}^{\infty}$ is both a continuous orthogonal, i.e.,

$$\frac{1}{h}\int_{\mathbb{R}} S(j,h)\circ(x) S(k,h)\circ(x)\,dx = \delta_{j,k}$$

and a discrete orthogonal, i.e.,

$$S(j,h)\circ(kh) = \delta_{j,k}, \quad j,k \in \mathbb{Z}$$

basis.

In the next two theorems, we shall use the notation $|(\cdot)|$ to denote $\sup_{\ell\in\mathbb{Z}}(\cdot)$, and $\|\cdot\|$ to denote $\sup_{x\in\Gamma}|(\cdot)|$, with $\Gamma=\mathbb{R}$ for the case of Theorem 2.2, and with Γ defined as in Section 2.3 for the case of Theorem 2.3.

Theorem 2.2 (Stenger [18, Ch. 3]). *If $f\in H^1(\mathscr{D}_d)$, if $h>0$, j is any fixed nonnegative integer, and if $\ell\in\mathbb{Z}$, then, as $h\to 0$:*

$$
\begin{gathered}
\left\| f(x) - \sum_{k=-\infty}^{\infty} S(k,h)\circ(x)\, f(kh) \right\| = \mathcal{O}(\mathrm{e}^{-\pi d/h}),\\
\sup_{x\in(-\pi/h,\pi/h)} \left| \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}xt} f(t)\,\mathrm{d}t - h \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}khx} f(kh) \right| = \mathcal{O}(\mathrm{e}^{-\pi d/h}),\\
h^j \left| f^{(j)}(\ell h) - h^{-j} \sum_{k=-\infty}^{\infty} \delta^{(j)}_{\ell-k} f(kh) \right| = \mathcal{O}(\mathrm{e}^{-\pi d/h}),\\
\left| \int_{-\infty}^{\ell h} f(x)\,\mathrm{d}x - h \sum_{k=-\infty}^{\infty} \delta^{(-1)}_{\ell-k} f(kh) \right| = \mathcal{O}(\mathrm{e}^{-\pi d/h}),\\
\left\| \frac{\mathrm{P.V.}}{\pi} \int_{\mathbb{R}} \frac{f(t)}{t-x}\,\mathrm{d}t - \sum_{k=-\infty}^{\infty} \frac{\cos[\pi(x-kh)/h]-1}{\pi(x-kh)/h} f(kh) \right\| = \mathcal{O}(\mathrm{e}^{-\pi d/h}),\\
\frac{1}{h}\int_{\mathbb{R}} f(t) S(j,h)\circ(t)\,\mathrm{d}t - f(jh) = \mathcal{O}(\mathrm{e}^{-\pi d/h}),
\end{gathered} \tag{2.5}
$$

where for $\ell\in\mathbb{Z}$,

$$
\begin{aligned}
\delta^{(j)}_{\ell} &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j} \operatorname{sinc}(x)\Big|_{x=\ell},\\
\delta^{(-1)}_{\ell} &= \int_{-\infty}^{\ell} \operatorname{sinc}(x)\,\mathrm{d}x.
\end{aligned} \tag{2.6}
$$

We may note, that if h is replaced by $\frac{1}{2}h$ in these formulas, then we get (roughly) twice as many significant figures of accuracy. Thus, when using these formulas, we can have the computer determine when a particular accuracy has been achieved. For example, when the difference between a Sinc approximation with step size h and one with step size $\frac{1}{2}h$ differs by $\varepsilon^{1/2}$, then the actual error in the $\frac{1}{2}h$ approximation is (roughly) of the order of ε.

2.3. Approximation on Γ = an interval or contour

Eq. (2.5) gives us a glimpse of the power of Sinc approximation on the real line $\mathbb{R}$. Suppose, next, that we have an approximation problem over an interval, or contour Γ. In that case we seek

a *one–one* transformation φ of Γ to $\mathbb{R}$. If we can furthermore select φ such that:
(i) With $f \in H^1(\mathscr{D})$, φ provides a conformal map of $\mathscr{D}$ to $\mathscr{D}_d$ with $\mathscr{D}_d$ defined as in (2.4);
(ii) We have

$$\Gamma \subset \mathscr{D}, \qquad \varphi(\Gamma) = R$$

and $\varphi(a) = -\infty$, $\varphi(b) = \infty$.

Theorem 2.3 (Stenger [18, Ch. 4]). *If these conditions are satisfied, then, letting $k \in \mathbb{Z}$, j a positive integer, and denoting the* Sinc points z_k *by*

$$z_k = \varphi^{-1}(kh) \tag{2.7}$$

we have, as $h \to 0$, that

$$\begin{aligned}
&\left\| \frac{f(x)}{\phi'(x)} - \sum_{k=-\infty}^{\infty} S(k,h) \circ \phi(x) \frac{f(z_k)}{\phi'(z_k)} \right\| = \mathscr{O}(\mathrm{e}^{-\pi d/h}),\\
&\left| \frac{h^j}{(\varphi'(z_\ell))^{j+1}} \left\{ f^{(j)}(z_\ell) - \sum_{k=-\infty}^{\infty} (S(k,h) \circ (\varphi(z_\ell))^{(j)} f(z_k) \right\} \right| = \mathscr{O}(\mathrm{e}^{-\pi d/h}),\\
&\sup_{\ell \in \mathbb{Z}} \left| \int_a^{z_\ell} f(x)\,\mathrm{d}x - h \sum_{k=-\infty}^{\infty} \delta_{\ell-k}^{(-1)} \frac{f(z_k)}{\varphi'(z_k)} \right| = \mathscr{O}(\mathrm{e}^{-\pi d/h}),\\
&\left\| \frac{\mathrm{P.V.}}{\pi} \int_\Gamma \frac{f(t)}{t-x}\,\mathrm{d}t - \sum_{k=-\infty}^{\infty} \frac{\cos\{\pi[\varphi(x) - kh]/h\} - 1}{\pi \varphi'(z_k)(x - z_k)/h} f(z_k) \right\| = \mathscr{O}(\mathrm{e}^{-\pi d/h}),\\
&\frac{1}{h} \int_\Gamma f(t) S(j,h) \circ \varphi(t)\,\mathrm{d}t - \frac{f(z_j)}{\varphi'(z_j)} = \mathscr{O}(\mathrm{e}^{-\pi d/h}).
\end{aligned} \tag{2.8}$$

2.4. The transformations

For example, if $\Gamma = (0,1)$, then we can take $\phi(z) = \log(z/(1-z))f$, in which case $\mathscr{D} = \{z \in \mathbb{C}\colon |\arg(z/(1-z))| < d\}$; if $\Gamma = (0,\infty)$, we can take $\phi(z) = \log(z)$, in which case $\mathscr{D} = \{z \in \mathbb{C}\colon \arg(z)| < d\}$, or $\log(\sinh(z))$, in which case $\mathscr{D} = \{z \in \mathbb{C}\colon \arg(\sinh(z)| < d\})$; if $\Gamma = \mathbb{R}$, then we can take $\phi(z) = z$, in which case $\mathscr{D} = \mathscr{D}_d$, or $\phi(z) = \log[z + (1+z^2)^{1/2}]$, in which case $\mathscr{D} = \{z \in \mathbb{C}\colon |\arg[z + (1+z^2)^{1/2}]| < d\}$, or $\phi(z) = \log\{\sinh[z + (1+z^2)^{1/2}]\}$, in which case $\mathscr{D} = \{z \in \mathbb{C}\colon |\arg\{\sinh[z + (1+z^2)^{1/2}]\}| < d\}$; and if Γ is the arc$\{z \in \mathbb{C}\colon z = \mathrm{e}^{\mathrm{i}\theta}, u < \theta < v\}$, where $0 < v - u < 2\pi$, we can take $\phi(z) = \mathrm{i}(v-u)/2 + \log[(z - \mathrm{e}^{\mathrm{i}u})/(\mathrm{e}^{\mathrm{i}v} - z)]$, in which case $\mathscr{D} = \{z \in C\colon |(v-u)/2 + \arg[(z - \mathrm{e}^{\mathrm{i}u})/(\mathrm{e}^{\mathrm{i}v} - z)| < d\}$.

We can usually remove the ambiguity in the above multiple choice cases, by considering a problem in view of the Sinc spaces which are described in Section 2.5. Many such considerations are illustrated in [18]. We illustrate this point below, for the case of Sinc quadrature.

2.5. Sinc spaces

Although the approximations (2.5) and (2.8) are very accurate, they still have one undesirable feature, namely, that we require the evaluation of an infinite series. For example, if we take $f(x) = 1/(1+x^2)$ in the first of (2.6), then the infinite sum is accurate, since we then have $d = 1$, so that the error is $\mathcal{O}(e^{-\pi/h})$. On the other hand, we end up summing a slowly convergent series, so that the complexity to achieve an error of the order of ε is $\mathcal{O}(1/\varepsilon \log(1/\varepsilon))$. We are, thus, motivated to define the *Sinc spaces,* which will enable us to truncate each of the above infinite series in such a way, that complexity of evaluating them is $\mathcal{O}((\log(1/\varepsilon))^2)$.

Let us now consider some simple Sinc spaces which go hand in hand with Sinc methods. Let $\alpha \in (0,1]$, $\beta \in (0,1]$, and $0<d<\pi$. Let $M_{\alpha,\beta}(\mathscr{D})$ denote the family of all functions $f \in \mathrm{Hol}(\mathscr{D})$, which have finite limits $f(a)$ and $f(b)$ at the end points a and b of Γ, such that if we set

$$\rho(z) = \exp(\phi(z)) \tag{2.9}$$

then

$$\begin{aligned} f(z) - f(a) &= \mathcal{O}(|\rho(z)|^{\alpha}) \quad \text{as } z \to a, \\ f(z) - f(b) &= \mathcal{O}(|\rho(z)|^{-\beta}) \quad \text{as } z \to b. \end{aligned} \tag{2.10}$$

where the limits are taken with respect to z from within $\mathscr{D}$. The notation $L_{\alpha,\beta}(\mathscr{D})$ will be used for that subset of functions $f \in M_{\alpha,\beta}(\mathscr{D})$ for which $f(a) = f(b) = 0$. These Sinc spaces (and their appropriate extension to more than one dimension, as described in e.g. [18, Sections 6.5 or 6.6]) house nearly all solutions to differential and integral equations in applications.

The most important properties of these spaces are, in essence, that if $f \in M_{\alpha,\beta}(\mathscr{D})$, then [14, 18, Section 4.1] $f'/\phi' \in L_{\alpha,\beta}(\mathscr{D})$; if $f \in M_{\alpha,\beta}(\mathscr{D})$, and if $(1/\phi')'$ is uniformly bounded in $\mathscr{D}$ then $f^{(n)}/(\phi')^n \in L_{\alpha,\beta}(\mathscr{D})$, $n = 1,2,3,\ldots$. Conversely, if $f'/\phi' \in L_{\alpha,\beta}(\mathscr{D})$, then $f \in M_{\alpha,\beta}(\mathscr{D})$. For example, if (a,b) is a finite interval, if f is of class Lip_α on [a, b], and if f is analytic in some region containing (a,b), then $f \in M_{\alpha,\alpha}(\mathscr{D})$.

2.6. Bases and notations

Let N denote a positive integer, and let integers M, and m, a diagonal matrix $D(u)$ and an operator V be defined as follows:

$$\begin{aligned} &N = \text{positive integer}, \\ &M = [\beta N/\alpha], \\ &m = M + N + 1, \\ &D(u) = \mathrm{diag}[u(z_{-M}), \ldots, u(z_N)] \end{aligned} \tag{2.11}$$

and

$$V(u) = (u(z_{-M}), \ldots, u(z_N))^{\mathrm{T}},$$

where $[\cdot]$ denotes the greatest integer function, u is an arbitrary function defined on Γ, and T denotes the transpose. We set

$$\begin{aligned}
&\mathrm{sinc}(z) = \frac{\sin(\pi z)}{\pi z},\\
&h = \left(\frac{\pi d}{\beta N}\right)^{1/2},\\
&z_j = \varphi^{-1}(jh), \quad j \in \mathbb{Z}\\
&\gamma_j = \mathrm{sinc}\{[\varphi - jh]/h\}, \quad j = -M, \ldots, N,\\
&w_j = \gamma_j, \quad j = -M+1, \ldots, N-1,\\
&w_{-M} = \frac{1}{1+\rho} - \sum_{j=-M+1}^{N} \frac{1}{1+\mathrm{e}^{jh}}\gamma_j,\\
&w_N = \frac{\rho}{1+\rho} - \sum_{j=-M}^{N-1} \frac{1}{1+\mathrm{e}^{-jh}}\gamma_j,\\
&\varepsilon_N = N^{1/2}\mathrm{e}^{-(\pi d\beta N)^{1/2}}.
\end{aligned} \tag{2.12}$$

We remark that in the special when we know a priori that f vanishes at the left endpoint a of Γ, then it is more convenient to take $w_{-M} = \gamma_{-M}$, and similarly we take $w_N = \gamma_N$ for the case when f vanishes at the right endpoint b of Γ.

Let us also define a *row vector of basis functions*,

$$w(z) = (w_{-M}(z), \ldots, w_N(z)) \tag{2.13}$$

with w_j defined as in (2.12).

We shall also define a norm by

$$\|f\| = \sup_{x\in\Gamma} |f(x)|. \tag{2.14}$$

2.7. Symbolic and matrix Sinc approximation

The following approximations are possible via Sinc methods, each having error ε_N, with ε_N defined in (2.12) above.

To achieve this ε_N error, we assume $f \in M_{\alpha,\beta}(\mathscr{D})$ for the case of interpolation and differentiation, and we assume that $f/\phi' \in L_{\alpha,\beta}(\mathscr{D})$ for the case of definite and indefinite integration. For the case of Sinc convolution, we assume, in essence, that the convolution of f and g belongs to $M_{\alpha,\beta}(\mathscr{D})$.

- *Sinc interpolation*: This basic procedure is described in the following theorem.

Theorem 2.4 (Kowalski et al. [14] and Stenger [18, Section 4.1]). *If $f \in M_{\alpha,\beta}(\mathscr{D})$, then as $N \to \infty$,*

$$\|f - wVf\| = \mathcal{O}(\varepsilon_N). \tag{2.15}$$

For Sinc interpolation, we may thus associate the unit matrix, I of order m, i.e.,

$$Vf = IVf. \tag{2.16}$$

- *Sinc collocation*: While the above-defined Sinc basis completely determines the formulas of Sinc approximation, the role of the basis is primarily in the evaluation, or plotting of the final solution. This is the point of the theorem of this section, which enables us to reduce all Sinc computations to the solution of (linear or nonlinear) algebraic equations.

Theorem 2.5 (Kowalski et al. [14]). *If the conditions of Theorem 2.4 are satisfied, and if $c = (c_{-M}, \ldots, c_N)^{\mathrm{T}}$ is a vector of complex numbers, such that*

$$\left(\sum_{j=-M}^{N} |f(z_j) - c_j|^2 \right)^{1/2} < \delta_N, \tag{2.17}$$

where δ_N is a positive number, depending on N, then, as $N \to \infty$,

$$\|f - w \cdot c\| = \mathcal{O}(\varepsilon_N + \delta_N). \tag{2.18}$$

- *Sinc differentiation*: One form of Sinc differentiation is described in the following theorem.

Theorem 2.6 (Kowalski et al. [14] and Stenger [18, Section 4.4]). *If $f \in M_{\alpha,\beta}(\mathscr{D})$, and if $(1/\varphi')'$ is uniformly bounded in $\mathscr{D}$, then, for $k = 1, 2, \ldots,$ as $N \to \infty$,*

$$\left\| h^k \frac{f^{(k)}}{(\varphi')^k} - wD(h^k/(\varphi')^k)\,[w_j^{(k)}(z_i)]\,Vf \right\| = \mathcal{O}(\varepsilon_N). \tag{2.19}$$

In this case, defining a matrix A of order m as a product of a diagonal matrix $D((h/\varphi'(z_i))^k)$ and a square matrix of order m with (i,j)th element $\gamma_j^{(k)}(z_i)$, Sinc differentiation in matrix notation takes the form

$$D((h/\varphi')^k)\,Vf^{(k)} \approx A\,Vf. \tag{2.20}$$

The accuracy as a function of m depends on the nature of the singularity, although for a problem without singularities (i.e., for the case when $\alpha = \beta = 1$, $m = 15$ yields about 4 places of accuracy, and $m = 30$ about 5. Indeed, these values of the error as a function of m hold for all of the Sinc methods. In the case when we know an estimate of the nature of the singularity is known, we can thus obtain an a priori estimate on the size of m based on the fact that the error is roughly proportional to $\exp\{-(\pi d\gamma N)^{1/2}\}$, with $\gamma = \min(\alpha, \beta)$.

We remark here that the *Toeplitz matrices*

$$I^{(k)} = [\delta_{i-j}^{(k)}] \tag{2.21}$$

with $\delta_j^{(k)}$ defined as in (2.6) arise repeatedly in the expression (2.19). Relatively, little is known about these matrices. It may be shown, e.g. (see [18, pp. 477–478]), that every eigenvalue λ of the matrix $I^{(2)}$ of order m satisfies the inequality

$$-\pi^2 < \lambda < \frac{-\pi^2}{(m+1)^2}, \tag{2.22}$$

and this fact enables us to easily deduce that the systems of equations arising when solving PDEs by Sinc methods are generally well conditioned. If k and m are both odd integers, then $I^{(k)}$ has a zero determinant, since in that case $I^{(k)}$ is a skew symmetric matrix of odd order. On the other hand, the determinant of $I^{(k)}$ is nonzero if either k is even (in which case $I^{(k)}$ is a negative definite symmetric Toeplitz matrix) or else if k is odd and m is even (in which case the skew symmetric Toeplitz matrix $I^{(k)}$ is of even order). It is easy to obtain an estimate of the largest eigenvalue of these matrices, whereas, there does not appear to be an effective technique for accurately estimating the magnitude of the smallest eigenvalue of $I^{(k)}$ in the case when k is odd and m is even. In particular, it would be nice to know this for the case of the even order matrix $I^{(1)}$, which is useful for solving first-order initial value problems for ordinary differential equations [10].

• *Sinc quadrature*: This simple Sinc approximation is described as follows.

Theorem 2.7 (Kowalski et al. [14] and Stenger [18, Section 4.2]). *If $f/\varphi' \in L_{\alpha,\beta}(\mathscr{D})$, then, as $N \to \infty$,*

$$\left| \int_\Gamma f(x)\,\mathrm{d}x - (V(h/\varphi'))^{\mathrm{T}}\, Vf \right| = \mathcal{O}(\varepsilon_N). \tag{2.23}$$

Notice the linear algebraic "inner product" form of Sinc quadrature.

For example, let us return to the approximation of the integral $J \equiv \int_{\mathbb{R}} f(x)\,\mathrm{d}x$, with $f(x) = (1+x^2)^{-1}$, which was mentioned at the outset of Section 2.5. We now illustrate some more efficient methods of evaluating this integral, in view of the above definition of Sinc spaces.

1. If we take the identity map, $\varphi(x) = x$, then $\mathscr{D}$ is the strip $\mathscr{D}_d$ with $0 < d < 1$, and in this case, $\varphi'(x) = 1$, and $\rho(x) = \exp(\varphi(x)) = \mathrm{e}^x$. In this case, $|f(x)/\varphi'(x)| \sim |x|^{-2}$ as $x \to \pm\infty$, whereas $\rho(x)^\alpha = \exp(\alpha x)$ (for consideration of x in a neighborhood of $-\infty$, and $\rho(x)^{-\beta} = \exp(-\beta x)$ (for consideration of x in a neighborhood of ∞). Hence, the conditions of Theorem 2.7 cannot be satisfied.
2. If we take the map $\varphi(x) = \log(x + \sqrt{1+x^2})$, then $\mathscr{D}$ is the "hour-glass" shaped region given by $\{z \in \mathbb{C}\colon z = \sinh(w), w \in \mathscr{D}_d\}$, with $d \in (0, \frac{1}{2}\pi)$. In this case, $\varphi'(x) = 1/\sqrt{1+x^2} \sim 1/|x|$ as $x \to \pm\infty$, $\rho(x) \sim 1/(2|x|)$, $x \to -\infty$, and $\rho(x) \sim 2x$, $x \to \infty$. Hence, the conditions of Theorem 2.7 can be satisfied with $\alpha = \beta = 1$, $0 < d < \frac{1}{2}\pi$, and by taking $N, M (M = N)$ h, z_j and ε_N as in (2.12) we get
$$J - h \sum_{j=-N}^{N} \cosh(jh) f(\sinh(jh)) = \mathcal{O}(\varepsilon_N).$$
3. We could, of course, split the integral J as an integral from $-\infty$ to 0 plus an integral from 0 to ∞. Since the integration over $\Gamma = (0, \infty)$ yields $(\frac{1}{2})J$, we could take $\varphi(x) = \log(x)$, in which case $\varphi'(x) = 1/x$, and $\mathscr{D}$ is the sector $\{z \in \mathbb{C}\colon |\arg(z)| < d\}$, $0 < d < \pi\frac{1}{2}$. We now have $f(x) \sim 1$ as $x \to 0$, and (as above) $f(x) \sim x^{-2}$ as $x \to \infty$. Hence, the conditions of Theorem 2.7 can be met

with $\alpha=\beta=1$, $0<d<\frac{1}{2}\pi$, to get

$$J - 2h\sum_{j=-N}^{N} e^{jh} f(e^{jh}) = \mathcal{O}(\varepsilon_N)$$

as $N \to \infty$.

This latter map, $\log(\cdot)$, could, of course, be used to accurately approximate a more difficult to evaluate integral, such as, e.g.,

$$\int_0^\infty \frac{x^{-2/3}\log(x)\arctan(\sqrt{1+x^2})}{1+x^2}\,dx.$$

- *Sinc indefinite integration*: For the case of Sinc indefinite integration and Sinc convolution, we determine matrices A and B of order m as follows. Corresponding to an arbitrary integer ℓ, we define $\delta_\ell^{(-1)}$ as in (2.6), and we form a matrix $I^{(-1)}=[\delta_{i-j}^{(-1)}]$, with $\delta_{i-j}^{(-1)}$ denoting the (i,j)th element of $I^{(-1)}$, and we then set $A=hI^{(-1)}D(1/\phi')$, $B=h(I^{(-1)})^{\mathrm{T}}D(1/\phi')$.

Theorem 2.8 (Kowalski et al. [14] and Stenger [18, Section 4.5]). *If* $f/\varphi'\in L_{\alpha,\beta}(\mathscr{D})$, *then, as* $N\to\infty$,

$$\begin{aligned}\left\|\int_a^x f(t)\,dt - (wAVf)(x)\right\| &= \mathcal{O}(\varepsilon_N),\\ \left\|\int_x^b f(t)\,dt - (wBVf)(x)\right\| &= \mathcal{O}(\varepsilon_N).\end{aligned} \tag{2.24}$$

In matrix notation, indefinite integration thus takes the form

$$\begin{aligned}V\int_a^x f(t)\,dt &\approx AVf,\\ V\int_x^b f(t)\,dt &\approx BVf.\end{aligned} \tag{2.25}$$

These equations have been used to develop a computer package for solving ordinary differential equation initial value problems [23].

- *Sinc convolution*: For the case of Sinc convolution, we define matrices A and B as for Sinc indefinite integration above, and in addition we shall also require the "Laplace transform",

$$F(s) = \int_E e^{-t/s} f(t)\,dt, \quad E\supseteq(0,(b-a)), \tag{2.26}$$

where the quotes indicate that we have replaced the s in the usual Laplace transform with $1/s$. Our model integrals to be collocated are

$$\begin{aligned}p(x) &= \int_a^x f(x-t)\,g(t)\,dt,\\ q(x) &= \int_x^b f(t-x)\,g(t)\,dt,\end{aligned} \tag{2.27}$$

where $x\in\Gamma$.

In presenting these convolution results, we shall assume that $\Gamma=(a,b)\subseteq\mathbb{R}$. For Sinc collocation of p and q, we assume that:

1. The "Laplace transform" (2.26) exists for all s such that $\Re s>0$; and (roughly – see [19] for more precise assumptions) that
2. p and q belong to $M_{\alpha,\beta}(\mathscr{D})$.

Theorem 2.9 (Kowalski et al. [14] and Stenger [18, Section 4.6]). *If the above conditions are satisfied, and if A and B are defined as in* (2.24) *above, then, as $N\to\infty$,*

$$\begin{aligned}\|p-wF(A)Vg\|&=\mathscr{O}(\varepsilon_N),\\ \|q-wF(B)Vg\|&=\mathscr{O}(\varepsilon_N).\end{aligned}\tag{2.28}$$

The matrix equivalents of these two expressions are

$$\begin{aligned}Vp&\approx F(A)Vg,\\ Vq&\approx F(B)Vg.\end{aligned}\tag{2.29}$$

As we shall see below, this expression yields powerful methods of solving differential equations in one and several dimensions, in the cases when such solutions can be expressed in terms of convolution-type integrals. Indeed, we have already solved such equations in [22, 16].

However, the solution of the following problems would be desirable, computational standpoint.

1. For example, one way of evaluating $F(A)c$ with c a column vector of order m is to first diagonalize A, i.e., to find the matrix of eigenvalues $S\equiv\operatorname{diag}[s_{-M},\ldots,s_N]$ of A, as well as the corresponding matrix X of eigenvectors, such that $A=XSX^{-1}$. It is not known whether A is always diagonalizable, or what the most efficient algorithm is for getting S and X. Once we know S and X, we can solve for the vector $e\equiv F(A)c$ by first solving the equation $Xb=c$ for b by LU decomposition, and then directly evaluating the product $e=XF(S)b$.
2. Since the domain of F is the open right-half plane, we want $\Re s_i>0$ for $i=-M,\ldots,N$. It is easily seen that this is the case for Γ an interval on $\mathbb{R}$ if and only if each eigenvalue of the matrix $I^{(-1)}$ defined as for Theorem 2.8 above lies in the open right half-plane (see [18, p. 228]). Recently, Naghsh–Nilchi (see e.g., [16]) numerically evaluated all of the eigenvalues of each matrix $I^{(-1)}$ of orders 2–513, to find that the eigenvalues of all of these matrices do, indeed, lie in the open right half-plane. It is not known, however, whether or not this is always the case; the author of the present paper offers $200.00 to the first person to prove or disprove this result.
3. The only connection with "Sinc" that the above *Sinc convolution* procedure has is the Sinc indefinite integral matrix, i.e., it would be possible to use an indefinite integral matrix that has been constructed via any other basis. Fox example, one could use Chebyshev polynomials (as interpolating polynomials) for approximating indefinite convolutions over $[-1,1]$, or a basis such as $\{e^{-x}L_n(x)\}_{n=0}^{\infty}$, with L_n the Laguerre polynomial for approximating indefinite convolutions over $(0,\infty)$.
4. The process of matrix diagonalization that is required for Sinc convolution involves a lot of work, and big payoffs occur only in more than one dimension. With reference to [2, 3], Schur forms are easier to obtain, and an efficient algorithm for the exponential of a matrix based on Schur may be more efficient than diagonalization.

3. Sinc solution of a Poisson problem

We illustrate here an extension of the above Sinc convolution procedure to the approximate solution of a Poisson problem in three dimensions. The above one-dimensional convolution approximations can be readily applied to evaluate convolution integrals in more than one dimension. Suppose, for sake of illustration, that we wish to evaluate the following three-dimensional convolution integral expression of a solution to the Poisson problem of the form $u_{xx}+u_{yy}+u_{zz}=-g$ in $V=(a_1,b_1)\times(a_2,b_2)\times(a_3,b_3)$, given by

$$\Psi(x,y,z)=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\frac{g(\xi,\eta,\zeta)}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \tag{3.1}$$

for $(x,y,z)\in V$.

In order to solve this problem, we require the three dimensional "Laplace transform" $F(u,v,w)$ of the convolution kernel, which can, in fact, be explicitly obtained, and is given by

$$\begin{aligned}F(u,v,w)&=\int_0^\infty\int_0^\infty\int_0^\infty\frac{\exp\{-x/u-y/v-z/w\}}{4\pi\sqrt{x^2+y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\&=\left(\frac{1}{u^2}+\frac{1}{v^2}+\frac{1}{w^2}\right)^{-1}\{-\tfrac{1}{8}+G(u,v,w)+G(v,w,u)+G(w,u,v)\},\end{aligned} \tag{3.2}$$

where, setting $\lambda=\sqrt{v^2+w^2}$,

$$G(u,v,w)=\frac{vw}{8\pi u\lambda}\ln\left(\frac{(\lambda+v)(\lambda+w)}{(\lambda-v)(\lambda-w)}\right). \tag{3.3}$$

The multidimensional convolution integral (3.1) can be readily split into 8 indefinite convolution integrals, such as

$$\Psi^{(1)}(x,y,z)=\int_{a_1}^{x}\int_{y}^{b_2}\int_{a_3}^{z}\frac{g(\xi,\eta,\zeta)}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}\xi. \tag{3.4}$$

To evaluate the eight integrals of the form $\Psi^{(1)}$ at all of the Sinc points $\{(z_i^{(1)},z_j^{(2)},z_k^{(3)})$: $i=-M_1,\ldots,N_1$, $j=-M_2,\ldots,N$, $k=-M_3,\ldots,N_3\}$, we require 3 transformations φ_i: $(a_i,b_i)\to\mathbb{R}$, $i=1,2,3$. We then determine h_i and the Sinc points $z_j^{(i)}$ as described above, we evaluate the functions $\varphi_i'(z_j^{(i)})$, and we form the matrices

$$\begin{aligned}&A_i=h_i I_{m_i}^{(-1)}D(1/\phi_i')=X_i S_i X_i^{-1},\ i=1,3,\\&A_2=h_2(I_{m_2}^{(-1)})^{\mathrm{T}}D(1/\phi_2')=X_2 S_2 X_2^{-1},\end{aligned} \tag{3.5}$$

where $m_i=M_i+N_i+1$, each $S_i=\mathrm{diag}[s_{-M_i}^{(i)},\ldots,s_{N_i}^{(i)}]$ is a diagonal matrix of eigenvalues of the matrix A_i, and X_i is the corresponding matrix of eigenvectors. We then evaluate the array $g_{ijk}=g(z_i^{(1)},z_j^{(2)},z_k^{(3)})$, and we form a vector g in which the subscripts appear in the order (call it lexicographic) dictated by the order of appearance of the subscripts in the Fortran do loop, "DO $k=-M_3,N_3$", followed by "DO $j=-M_2,N_2$", followed by "DO $i=-M_1,N_1$". We then also form the diagonal matrix F in which the entries are the values $F_{ijk}=F(s_i^{(1)},s_j^{(2)},s_k^{(3)})$, with the function F and the eigenvalues $s_j^{(i)}$

defined as above, and where we also list the values F_{ijk} in the same lexicographic order as for g_{ijk}. Then, similarly forming an array $\Psi_{ijk}^{(1)}$, we can define a vector Ψ_1 by listing the elements $\Psi_{ijk}^{(1)}$ in lexicographic order. It can then be shown that if Ψ_1 is defined by the matrix (Kronecker) product

$$\Psi_1 = \Phi\, g,$$

$$\Phi = X_3 \otimes X_2 \otimes X_1\, F\, X_3^{-1} \otimes X_2^{-1} \otimes X_1^{-1},$$

then the corresponding numbers $\Psi_{ijk}^{(1)}$ are accurate approximations of the values $\Psi^{(1)}(z_i^{(1)}, z_j^{(2)}, z_k^{(3)})$.

We remark that, due to the Kronecker product representation of the matrix Φ the numerical determination of the vector $\Psi_1 = \Phi\, g$ can be carried out in parallel, without storage of the huge matrix Φ in this equation, which may be an asset, especially for problems in 3 or more dimensions.

The other seven indefinite integrals (which when added to Ψ_1 have sum Ψ) can similarly be dealt with. By adding up the resulting eight vectors Ψ_j, we get an accurate approximation to all of the values $\Psi(z_i^{(1)}, z_j^{(2)}, z_k^{(3)})$ defined in the definite convolution integral (3.1). Using Sinc interpolation, we can then get an almost equally accurate approximation to the function Ψ at all points of V. It may, moreover be shown, assuming that the function $g(\cdot, y, z)$, (and, additionally, making similar assumptions about the functions $g(x, \cdot, z)$ and $g(x, y, \cdot)$) is analytic on (a_1, b_1), for all $(y, z) \in [a_2, b_2] \times [a_3, b_3]$ that if e.g., $M_i = N_i = N$ for $i = 1, 2, 3$, then the uniform error of approximation is of the order of $\exp(-c N^{1/2})$, with c a constant that is independent of N.

Of course, the skeptical user may remark here that he/she can also efficiently solve the problem (3.1) with FFT. This may be true in certain circumstances. For example, if the region V is bounded, and if the solution is smooth on $\mathbb{R}^3$, with period $b_1 - a_1$ in the variable x, with period $b_2 - a_2$ in the variable y, and with period $b_3 - a_3$ in the variable z, then by using m points in each variable, we could achieve a rate of convergence $\mathcal{O}(\mathrm{e}^{-cm})$. One can also achieve an exponential rate of convergence via a multigrid method, if the solution is sufficiently smooth in a bounded region V. On the other hand, the time complexity of the above-described Sinc method on a sequential machine is of the order of $(\log(\varepsilon))^8$ even if one of the intervals has a finite end point, in the neighborhood of which g "blows up" at the rate $\mathcal{O}(r^{\alpha-2})$, as $r \to 0$, with α an arbitrary positive number, and with r being the distance to a boundary point, or if the region V is infinite. We know of no other algorithm, i.e. this efficient for small ε.

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